From a possibility theory view of formal concept analysis to the possibilistic handling of incomplete and uncertain contexts

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Outline

1. Formal Concept Analysis (FCA)
   - Basics
   - Attribute implications
2. Asymmetric composition of possibilistic operators
   - Possibilistic operators: possibility operator $(.)^{\Pi}$ and necessity operator $(.)^{\mathrm{N}}$
   - Asymmetric composition $(\mathrm{N} \circ \Pi)$
   - Complete lattice of $\mathrm{N}\Pi$-Pairs
3. Disjunctive attribute implications
4. Possible and certain implication in incomplete contexts
5. Implication from gradual uncertainty contexts
6. Conclusion

Formal concept analysis: formal context

- A formal context is defined as a set structure $\mathcal{K} := (\mathcal{O}, \mathcal{P}, \mathcal{R})$ for which $\mathcal{R}$ is a binary relation between some objects of $\mathcal{O}$ and some attributes of $\mathcal{P}$.
  Example:

$\mathcal{O}$={John,Maria,Peter,Clara},
$\mathcal{P}$={Man,Woman,Mother,Father, Parent},
$\mathcal{R}$={{John,Man}, {Maria,Woman},{Peter,Father},...}.

attributes

R	Man	Woman	Mother	Father	Parent
John	×				
Maria		×			
Peter	×			×	×
Clara		×	×		×

Objects

R: (Peter,Father)

Formal concept analysis: formal concept

- Formal concepts are induced using the Galois derivation operator $(.)^{\Delta}$.
- X^{Δ} corresponds to the set of attributes that are satisfied by all objects in X:

$$X^{\Delta} = \{ a \in \mathcal{P} \mid \forall x \in \mathcal{O} \ (x \in X \Rightarrow (x, a) \in \mathcal{R})\}$$
$$\{Peter, Clara\}^{\Delta} = \{Parent\}$$

- A^{Δ} corresponds to the set of objects that satisfy all attributes in A:

$$A^{\Delta} = \{ x \in \mathcal{O} \mid \forall a \in \mathcal{P} \ (a \in A \Rightarrow (x, a) \in \mathcal{R})\}$$
$$\{Parent\}^{\Delta} = \{Peter, Clara\}$$

- A pair $\langle\{X, A\}\rangle$ such that $X^{\Delta} = A$ and $A^{\Delta} = X$ is called **formal concept.**

$\langle\{\{Peter, Clara\}, Parent\}\rangle$ is a formal concept
{Peter,Clara} is called **extent**, {Parent} is called **intent.**

Formal concept analysis: attribute implications

- An attribute implications $A \to B$ holds in a formal context $\mathcal{K} := (\mathcal{O}, \mathcal{P}, \mathcal{R})$ if any object satisfying all elements of a set of attributes A also satisfies all elements of a set of attributes B.

$$\begin{aligned} & (A)^{\Delta} \subseteq (B)^{\Delta} \\ \Leftrightarrow \quad & B \subseteq (A)^{\Delta\Delta} \end{aligned}$$

R	Man	Woman	Mother	Father	Parent
John	×				
Maria		×			
Peter	×			×	×
Clara		×	×		×

{Mother} → {Woman}
{Father} → {Man}
{Man, Parent} → {Father}
{Woman, Parent} → {Mother}

Attribute implications

- It is important to remark that the underlying semantics is a *conjunctive* one

Asymmetric composition of possibilistic operators

- Taking lesson from possibility theory

  3 other power set derivation operators can be introduced [*Dubois*, *Dupin de Saint-Cyr*, & *Prade*, 2007]

  - The possibility operator (denoted $(.)^{\Pi}$)
  - The necessity operator (denoted $(.)^{N}$)
  - The dual sufficiency operator (denoted $(.)^{\nabla}$)

Asymmetric composition ($\mathrm{N} \circ \Pi$)

- $(A)^{\Pi}$ corresponds to the set of objects that are associated with at least one attribute in A

$$\begin{aligned}(A)^{\Pi} &= \{x \in \mathcal{O} \mid \exists a \in A, (x,a) \in \mathcal{R}\} \\ & \{\textit{Man}, \textit{Woman}\}^{\Pi} = \{\textit{John}, \textit{Maria}, \textit{Peter}, \textit{Clara}\}\end{aligned}$$

- $(A)^{\mathrm{N}}$ corresponds to the set of objects such that any attribute that is satisfied by one of them is necessarily in A

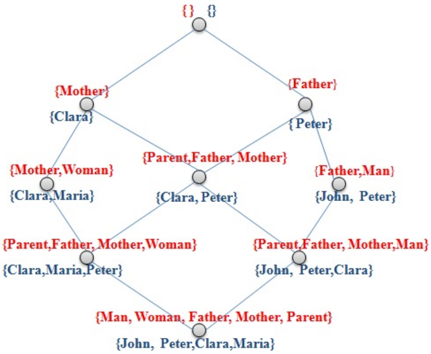
$$\begin{aligned}(A)^{\mathrm{N}} &= \{x \in \mathcal{O} \mid \forall a \in \mathcal{P}\ ((x,a) \in \mathcal{R} \Rightarrow a \in A)\} \\ & \{\textit{Parent}\}^{\mathrm{N}} = \{\textit{Clara}, \textit{Peter}\}\end{aligned}$$

- A pair $\langle X, A\rangle$ such that $X = A^{\Pi}$ and $A = X^{\mathrm{N}}$ is called an $\mathrm{N}\Pi$-pair (i.e., open-closed pair)

$\langle\{\textit{Peter}, \textit{Clara}\}, \{\textit{Father}, \textit{Mother}, \textit{Parent}\}\rangle$ is an $\mathrm{N}\Pi$-pair

Complete lattice of NΠ-Pairs

- $(\preccurlyeq)$ defines a partial order between two NΠ-pairs:
  $$\langle X_1, A_1 \rangle \preccurlyeq \langle X_2, A_2 \rangle \text{ iff } X_1 \subseteq X_2 \text{ (or } A_1 \subseteq A_2).$$
- The set of all NΠ-pairs with a partial order $\preccurlyeq$ forms a complete lattice ($\mathfrak{L}_{N\Pi}$)



lattice of NΠ-pairs

Disjunctive attribute implications

- The disjunctive attribute implication $a_1 \vee ... \vee a_n \rightarrow b_1 \vee ... \vee b_m$ (i.e., $\bigvee A \rightarrow \bigvee B$ with $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$) holds in a formal context $\mathcal{K}(\mathcal{O}, \mathcal{P}, \mathcal{R})$ if and only if every object that is never satisfied by each attribute from B is also never satisfied by each attribute from A

$$\forall x \in \mathcal{O}, \text{if } b_1 \nsubseteq \{x\}^{\Pi} \wedge ... \wedge b_m \nsubseteq \{x\}^{\Pi} \;\; \text{then} \;\; a_1 \nsubseteq \{x\}^{\Pi} \wedge ... \wedge a_n \nsubseteq \{x\}^{\Pi}$$

$$\Leftrightarrow \quad \forall x \in \mathcal{O}, \text{ if } B \subseteq \overline{\{x\}^{\Pi}} \text{ then } A \subseteq \overline{\{x\}^{\Pi}}$$

$$\Leftrightarrow \quad A \subseteq ((B)^{\Pi})^{N}$$

R	Man	Woman	Mother	Father	Parent
John	×				
Maria		×			
Peter	×			×	×
Clara		×	×		×

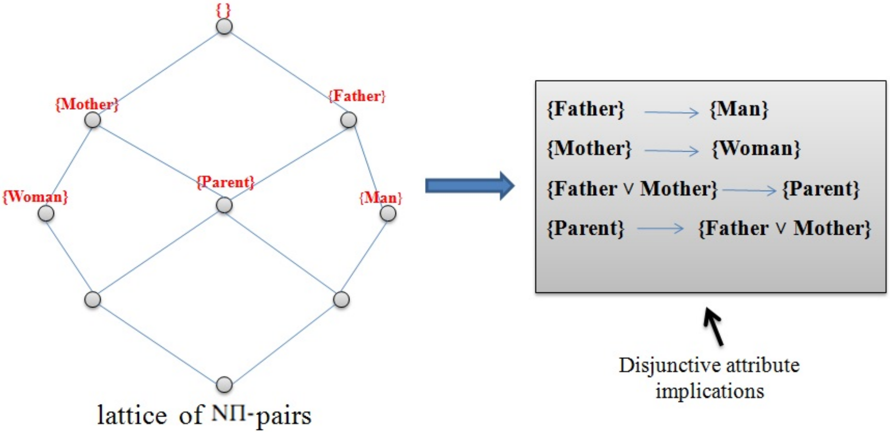
- $(\{Father, Mother\}^{\Pi})^{N} = \{Father, Mother, Parent\}$

  $\{Parent\} \subseteq (\{Father, Mother\}^{\Pi})^{N}$

  $Parent \rightarrow Father \vee Mother$

Disjunctive attribute implications

- The disjunctive attribute implications that hold in a formal context $\mathcal{K}(\mathcal{O},\mathcal{P},\mathcal{R})$ can be obtained from the lattice of $\mathrm{N\Pi}$-pairs.
  - $a \to \bigvee B$ holds in a formal context $\mathcal{K}(\mathcal{O},\mathcal{P},\mathcal{R})$ iff $(a^{\Pi},(a^{\Pi})^{\mathrm{N}}) \leq (B^{\Pi},(B^{\Pi})^{\mathrm{N}})$



Attribute implications & disjunctive attribute implications

- The disjunctive attribute implication $\bigvee A \rightarrow \bigvee B$ holds in formal context $\mathcal{K}(\mathcal{O}, \mathcal{P}, \mathcal{R})$ iff the attribute implication $\neg B \rightarrow \neg A$ holds in formal context $\overline{\mathcal{K}}(\mathcal{O}, \mathcal{P}, \overline{\mathcal{R}})$

A formal context $K(O,P,R)$

R	Man	Woman	Mother	Father	Parent
John	×				
Maria		×			
Peter	×			×	×
Clara		×	×		×

{Father} → {Man}

{Mother} → {Woman}

{Father ∨ Mother} → {Parent}

{Parent} → {Father ∨ Mother}

Disjunctive attribute implications

A formal context $\overline{K}(O,P,\bar{R})$

$\bar{R}$	¬Man	¬Woman	¬Mother	¬Father	¬Parent
John		×	×	×	×
Maria	×		×	×	×
Peter		×	×		
Clara	×			×	

{¬ Man} → {¬ Father}

{¬ Woman} → {¬ Mother}

{¬ Parent} → {¬ Father ∧ ¬ Mother}

{¬ Father ∧ ¬ Mother} → {¬ Parent}

Attribute implications

Incomplete formal context

- It is widely agreed that knowledge may be incomplete
- incomplete knowledge in FCA already studied by Burmeister & Holzer (2005) using Kleene 3-valued logic
- the incomplete formal context $\mathcal{K}_i(\mathcal{O}, \mathcal{P}, \{\times, -, ?\}, \mathcal{R}_i)$ consider the third value, denoted "?" :
  - $(x, a, \times) \in \mathcal{R}_i$: it is known that the object x has the attribute a
  - $(x, a, -) \in \mathcal{R}_i$: it is known that the object x does not have the attribute a
  - $(x, a, ?) \in \mathcal{R}_i$: it is unknown, whether the object x has the attribute a or not

R_i	a_1	a_2	a_3	a_4
x_1	×	×	×	?
x_2	×	×	×	×
x_3	?	_	×	?
x_4	_	×	_	×

Possible and certain implications in incomplete contexts

- From an incomplete formal context we obtain :
  - Certain attribute implications: if it holds in each formal context corresponding to a completion of $\mathcal{K}_j$.

$$\{a_1, a_2\} \to \{a_3\}$$

  - Possible attribute implications: if holds in at least one formal context corresponding to a completion of $\mathcal{K}_j$

$$\{a_2, a_3\} \to \{a_4\}$$

R_i	a_1	a_2	a_3	a_4
x_1	×	×	×	?
x_2	×	×	×	×
x_3	?	_	×	?
x_4	_	×	_	×

$K_*(O,P,R_*)$

R_*	a_1	a_2	a_3	a_4
x_1	×	×	×	_
x_2	×	×	×	×
x_3	_	_	×	_
x_4	_	×	_	×

...

Others formal context s

$K_j(O,P,R_j)$

$K^*(O,P,R^*)$

R^*	a_1	a_2	a_3	a_4
x_1	×	×	×	×
x_2	×	×	×	×
x_3	×	_	×	×
x_4	_	×	_	×

Possible and certain implications in incomplete contexts

- There exists exactly 2^n possible formal contexts $\mathcal{K}_j$ (n is the number of "?" in the initial incomplete formal context).
- An implication is certain if it is valid in each formal context $\mathcal{K}_j$; this condition may seem hard to verify at first glance.

Theorem 1

$$A \rightarrow B \text{ is a certain attribute implication in } \mathcal{K}_i \text{ iff } A^{\Delta}_{K^*} \subseteq B^{\Delta}_{K_*}$$

- Another problem is to determine a possible attribute implication that are holds in at least ont formal context $\mathcal{K}_j$, the following theorem facilitates this determination.

Theorem 2

$$A \rightarrow B \text{ is a possible attribute implication in } \mathcal{K}_i \text{ iff } A^{\Delta}_{K_*} \subseteq B^{\Delta}_{K^*}$$

Possible and certain disjunctive attribute implications in incomplete contexts

- In the case of conjunctive attribute implications we distinguish also :
  - Certain disjunctive attribute implications: if it holds in each formal context $\mathcal{K}_j$.

$$\{a_3\} \rightarrow \bigvee\{a_1, a_2\}$$

  - Possible disjunctive attribute implications: if it holds in at least ont formal context $\mathcal{K}_j$.

$$\{a_4\} \rightarrow \bigvee\{a_2, a_3\}$$

R_i	a_1	a_2	a_3	a_4	a_5
x_1	_	_	_	?	×
x_2	_	_	_	_	×
x_3	?	×	_	?	_
x_4	×	_	×	_	_

$K_*(O,P,R_*)$

R_*	a_1	a_2	a_3	a_4	a_5
x_1	_	_	_	_	×
x_2	_	_	_	_	×
x_3	_	×	_	_	_
x_4	×	_	×	_	_

...

Others formal context s

$K_j(O,P,R_j)$

$K^*(O,P,R^*)$

R^*	a_1	a_2	a_3	a_4	a_5
x_1	_	_	_	×	×
x_2	_	_	_	_	×
x_3	×	×	_	×	_
x_4	×	_	×	_	_

Possible and certain implications in incomplete contexts

- A disjunctive attribute implication is certain if it is valid in each formal context $\mathcal{K}_j$

Theorem 3

$\bigvee A \rightarrow \bigvee B$ is a certain disjunctive attribute implication iff $A^{\Pi}_{K^*} \subseteq B^{\Pi}_{K_*}$

- A disjunctive attribute implication is possible if it holds in at least ont formal context $\mathcal{K}_j$

Theorem 4

$\bigvee A \rightarrow \bigvee B$ is a possible disjunctive attribute implication iff $A^{\Pi}_{K_*} \subseteq B^{\Pi}_{K^*}$

Implications from gradual uncertainty contexts

- In an uncertain formal context the boxes are filled with pairs (α, β) of degrees of necessity
  - (α) is the necessity that the object has the attribute
  - (β) is the necessity that the object does not have the attribute

R	a_1	a_2	a_3	a_4
x_1	(1,0)	(0.7,0)	(0.8,0)	(0,0)
x_2	(1,0)	(0.9,0)	(1,0)	(1,0)
x_3	(0,0.5)	(0,0.3)	(1,0)	(0,0.5)
x_4	(0,0.7)	(0.7,0)	(0,1)	(1,0)

- Pairs (1,0) correspond to completely informed situations where it is known that object has the attribute (i.e., $+$)
- Pairs (0,1) correspond to completely informed situations where it is known that object does not have the attribute (i.e., $-$)
- Pairs (0,0) reflect total ignorance (i.e., ?)
- Pairs (α, β) s.t. $1 > \max(\alpha, \beta) > 0$ correspond to partial ignorance

Implications from gradual uncertainty contexts

- Consider a pair of thresholds (u, v) with $u > 0$ and $v > 0$. $\mathcal{K}_{(u,v)}$ is an incomplete formal context obtained by replacing:
  - all entries of the form $(\alpha, 0)$ such that $\alpha \geq u$ by (+)
  - all entries of the form $(\alpha, 0)$ such that $\alpha < u$ by (?)
  - all entries of the form $(0, \beta)$ such that $\beta \geq v$ by (-)
  - all entries of the form $(0, \beta)$ such that $\beta < v$ by (?)

R	a_1	a_2	a_3	a_4
x_1	(1,0)	(0.7,0)	(0.8,0)	(0,0)
x_2	(1,0)	(0.9,0)	(1,0)	(1,0)
x_3	(0,0.5)	(0,0.9)	(1,0)	(0,0.5)
x_4	(0,0.7)	(0.7,0)	(0,1)	(1,0)

→

$R_{(0.7,0.6)}$	a_1	a_2	a_3	a_4
x_1	×	×	×	?
x_2	×	×	×	×
x_3	?	_	×	?
x_4	_	×	_	×

$(K_{(0.7,0.6)})_*$

$(R_{(0.7,0.6)})_*$	a_1	a_2	a_3	a_4
x_1	×	×	×	_
x_2	×	×	×	×
x_3	_	_	×	_
x_4	_	×	_	×

...

Others formal context s

$(K_{(0.7,0.6)})^*$

$(R_{(0.7,0.6)})^*$	a_1	a_2	a_3	a_4
x_1	×	×	×	×
x_2	×	×	×	×
x_3	×	_	×	×
x_4	_	×	_	×

Conjunctive attribute implications from gradual uncertainty contexts

- $(\mathcal{K}_{(u,v)})_*$ does not depend on v, and increases when u decreases
- $(\mathcal{K}_{(u,v)})^*$ does not depend on u, and increases when v increases
- Observe that:
  - $A^{\Delta}_{(\mathcal{K}_{(u,v)})^*}$ increases when v increases
  - $B^{\Delta}_{(\mathcal{K}_{(u,v)})_*}$ decreases when u increases
- An attribute implication $A \to B$ is all the more certain as $A^{\Delta}_{(\mathcal{K}_{(u,v)})^*} \subseteq B^{\Delta}_{(\mathcal{K}_{(u,v)})_*}$ holds with u large and v large
  - The degree of certainty $\text{cert}(A \to B)$ of the attribute implication is equal to the maximum value w such that $A^{\Delta}_{(\mathcal{K}_{(w,w)})^*} \subseteq B^{\Delta}_{(\mathcal{K}_{(w,w)})_*}$
- A possibility degree can be attached to the attribute implication such that $A^{\Delta}_{(\mathcal{K}_{(u,v)})_*} \subseteq B^{\Delta}_{(\mathcal{K}_{(u,v)})^*}$ which is all the larger as u and v are larger

Disjunctive attribute implications from gradual uncertainty contexts

- The degree of certainty of disjunctive attribute implications $\text{cert}(\bigvee A \rightarrow \bigvee B)$ is equal to the maximum value w such that $A^{\Pi}_{(\mathcal{K}_{(w,w)})^*} \subseteq B^{\Pi}_{(\mathcal{K}_{(w,w)})_*}$
- A possibility degree is attached to attribute implication such that $A^{\Pi}_{(\mathcal{K}_{(u,v)})_*} \subseteq B^{\Pi}_{(\mathcal{K}_{(u,v)})^*}$ which is all the greater as u and v are greater

Conclusion

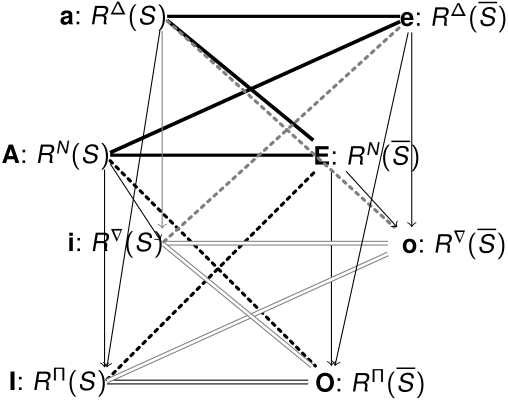
Contribution

- We show how to get disjunctive attribute implications
- The proposed approach considers "open-closed" pairs obtained by means of the asymmetric composition $(\mathrm{N} \circ \Pi)$ of necessity and possibility operators.
- We have generalized it to incomplete contexts and to gradual uncertainty contexts

Perspectives

- Interface with description logics
- We have only focused on composition $(.)^{\mathrm{N}\Pi}$.
  Further researches should concern the study of other possible compositions of possibilistic composite operators such that $(.)^{\Pi\Delta}$, $(.)^{\nabla\Delta}$, etc.

THANK YOU FOR YOUR ATTENTION



$$R^{N}(S) \cup R^{\Delta}(S) \subseteq R^{\Pi}(S) \cap R^{\nabla}(S)$$