Morphisms Between Pattern Structures and Their Impact on Concept Lattices

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Definition (adjunction)

Let $\mathbb{P} := (P, \leqslant)$ and $\mathbb{L} := (L, \leqslant)$ be posets,
and let $f : P \to L$ and $g : L \to P$ be maps.

$\mathbb{P} \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{L}$ or rather $(\mathbb{P}, \mathbb{L}, f, g)$ is a **poset adjunction**:

$$fx \leqslant y \Leftrightarrow x \leqslant gy \text{ for all } x \in P, y \in L.$$

Then f is called **residuated** from $\mathbb{P}$ to $\mathbb{L}$
and g is **residual** from $\mathbb{L}$ to $\mathbb{P}$.

Definition (kernel operater and closure operator)

Let $\mathbb{P} := (P, \leqslant)$ be a poset.

- Map $k : P \to P$ is **kernel operator** on $\mathbb{P}$:
  $kx \leqslant y \Leftrightarrow kx \leqslant ky$ for all $x, y \in P$.
- Subset K of P is **kernel system** in $\mathbb{P}$:
  Inclusion map $K \to P, x \mapsto x$ is residuated from $\mathbb{P}|K$ to $\mathbb{P}$.
- $c : P \to P$ **closure operator** on $\mathbb{P}$:
  $x \leqslant cy \Leftrightarrow cx \leqslant cy$ for all $x, y \in P$.
- Subset C of P is **closure system** in $\mathbb{P}$:
  Inclusion map $C \to P, x \mapsto x$ is residual from $\mathbb{P}$ to $\mathbb{P}|C$.
- Subset B of P is **bipolar system** in $\mathbb{P}$:
  B is kernel system and closure system in $\mathbb{P}$.

Theorem

$$\boxed{\textit{kernel operators on } \mathbb{P}} \overset{1-1}{\longleftrightarrow} \boxed{\textit{kernel systems in } \mathbb{P}}$$

$$k \longmapsto kP$$

$$\boxed{\textit{closure operators on } \mathbb{P}} \overset{1-1}{\longleftrightarrow} \boxed{\textit{closure systems in } \mathbb{P}}$$

$$c \longmapsto cP$$

Definition (pattern structure)

A triple $\mathcal{G} = (G, \mathbb{D}, \delta)$ is a **pattern setup** if G is a set, $\mathbb{D} = (D, \sqsubseteq)$ is a poset, and $\delta : G \to D$ is a map. In case every subset of $\delta G := \{\delta g \mid g \in G\}$ has an infimum in $\mathbb{D}$, we will refer to $\mathcal{G}$ as **pattern structure**.

A **projection** of $\mathcal{G}$ is defined as a kernel operator on $\mathbb{D}$.

Remark

If $\mathcal{G} = (G, \mathbb{D}, \delta)$ is a pattern structure then the set

$$\mathrm{C}_{\mathcal{G}} := \{\inf_{\mathbb{D}} \delta X \mid X \subseteq G\}$$

forms a closure system in $\mathbb{D}$.

Definition (pattern morphism)

If $\mathcal{G} = (G, \mathbb{D}, \delta)$ and $\mathcal{H} = (H, \mathbb{E}, \varepsilon)$ each is a pattern setup, then a pair (f, φ) forms a **pattern morphism** from $\mathcal{G}$ to $\mathcal{H}$ if $f : G \to H$ is a map and φ is a residual map from $\mathbb{D}$ to $\mathbb{E}$ satisfying $\varphi \circ \delta = \varepsilon \circ f$.

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Commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\;f\;} & H \\
\delta \big\downarrow & & \big\downarrow \varepsilon \\
\mathbb{D} & \xrightarrow[\;\varphi\;]{} & \mathbb{E}
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Question

Why is this a good concept of morphism for pattern structures?

Excursion to poset adjunctions and their morphisms.

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Definition

Let $\mathcal{P} := (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ and $\mathcal{Q} := (\mathbb{Q}, \mathbb{T}, \tau, \tau^+)$ be poset adjunctions. Then a pair (α, β) forms a **morphism** from $\mathcal{P}$ to $\mathcal{Q}$ if $(\mathbb{P}, \mathbb{Q}, \alpha, \alpha^+)$ and $(\mathbb{S}, \mathbb{T}, \beta, \beta^+)$ are poset adjunctions satisfying

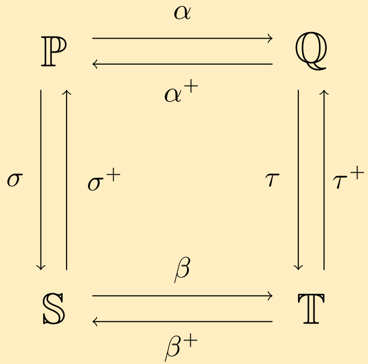
$$\tau \circ \alpha = \beta \circ \sigma$$

Remark: This implies $\alpha^+ \circ \tau^+ = \sigma^+ \circ \beta^+$, that is, the following diagrams are commutative:

Commutative diagrams

$$
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\alpha} & \mathbb{Q} \\
\sigma \downarrow & & \downarrow \tau \\
\mathbb{S} & \xrightarrow[\beta]{} & \mathbb{T}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{P} & \xleftarrow{\alpha^+} & \mathbb{Q} \\
\sigma^+ \uparrow & & \uparrow \tau^+ \\
\mathbb{S} & \xleftarrow[\beta^+]{} & \mathbb{T}
\end{array}
$$

Involved poset adjunctions



Definition (Concept Poset)

For a poset adjunction $\mathcal{P} = (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ let

$$\mathrm{B}\mathcal{P} := \{(p, s) \in P \times S \mid \sigma p = s \wedge \sigma^+ s = p\}$$

denote the set of (**formal**) **concepts** in $\mathcal{P}$. Then the **concept poset** of $\mathcal{P}$ is given by

$$\mathbb{B}\mathcal{P} := (\mathbb{P} \times \mathbb{S}) \mid \mathrm{B}\mathcal{P},$$

that is, $(p_0, s_0) \leqslant (p_1, s_1)$ holds iff $p_0 \leqslant p_1$ iff $s_0 \leqslant s_1$, for all $(p_0, s_0), (p_1, s_1) \in B\mathcal{P}$. If (p, s) is a formal concept in $\mathcal{P}$ then p is referred to as **extent** in $\mathcal{P}$ and s as **intent** in $\mathcal{P}$.

Theorem

Let (α, β) be a morphism from a poset adjunction $\mathcal{P} = (\mathbb{P}, \mathbb{S}, \sigma, \sigma^+)$ to a poset adjunction $\mathcal{Q} = (\mathbb{Q}, \mathbb{T}, \tau, \tau^+)$. Then

$$(\mathbb{B}\mathcal{P}, \mathbb{B}\mathcal{Q}, \Phi, \Psi)$$

is a poset adjunction for

$$\Phi : \mathrm{B}\mathcal{P} \to \mathrm{B}\mathcal{Q}, (p, s) \mapsto (\tau^+\beta s, \beta s)$$

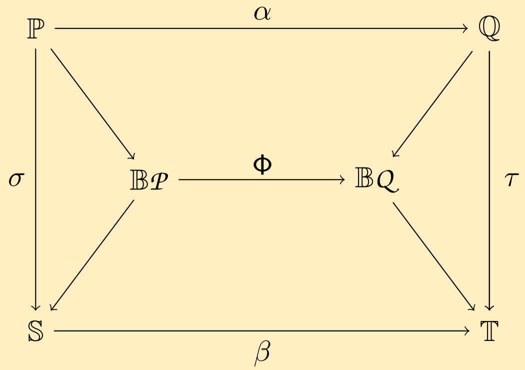
and

$$\Phi^+ : \mathrm{B}\mathcal{Q} \to \mathrm{B}\mathcal{P}, (q, t) \mapsto (\alpha^+ q, \sigma\alpha^+ q).$$

Remark

- If (p, s) is a formal concept in $\mathcal{P}$ then βs is an intent in $\mathcal{P}$.
- If in addition α is surjective then so is Φ.

Commutative diagram of residuated maps



Warning!

The map:

$$\mathbb{P} \times \mathbb{S} \to \mathbb{Q} \times \mathbb{T}, (p, s) \mapsto (\alpha p, \beta s)$$

is residuated, but its image of $\mathrm{B}\mathcal{P}$ is in general not contained in $\mathrm{B}\mathcal{Q}$.

Theorem

Under the conditions of the previous theorem the following hold:

(1) *If α is surjective then Φ is surjective too.*

(2) *If β is injective then Φ is injective too.*

(3) *If α is surjective and β is injective then Φ is an isomorphism from $\mathrm{B}\mathcal{P}$ to $\mathrm{B}\mathcal{Q}$..*

Proof.

(1) Assume that α is surjective, that is, $\alpha \circ \alpha^+ = id_Q$. Then for all $(p, s) \in \mathrm{B}\mathcal{P}$, the second component of $(\Phi \circ \Phi^+)(p, s)$ is $\beta\sigma\alpha^+ q = \tau\alpha\alpha^+ q = \tau q = s$. This yields $\Phi \circ \Phi^+ = id_{\mathrm{B}Q}$, that is, Φ is surjective.

(2) The first component of $(\Phi^+ \circ \Phi)(p, s)$ is $\alpha^+\tau + \beta s = \tau + \beta^+\beta s = \tau + s = p$. Therefore, $\Phi^+ \circ \Phi = id_{\mathrm{B}\mathcal{P}}$, which yields Φ being injective.

(3) If α is surjective and β is injective, than Φ and Φ^+ are naturally inverse by (1) and (2), that is, Φ is an isomorphism from $\mathrm{B}\mathcal{P}$ to $\mathrm{B}Q$.

□

Theorem

Let (f, φ) be a pattern morphism from a pattern structure $\mathcal{G} = (G, \mathbb{D}, \delta)$ to a pattern structure $\mathcal{H} = (H, \mathbb{E}, \varepsilon)$.
To apply the previous theorem we give the following construction: f gives rise to an adjunction (α, α^+) between the power set lattices $\mathbf{2}^G := (2^G, \subseteq)$ and $\mathbf{2}^H := (2^H, \subseteq)$ via

$$\alpha : 2^G \to 2^H, X \mapsto fX and \alpha^+ : 2^H \to 2^G, Y \mapsto f^{-1}Y.$$

and

$$\alpha^+ : 2^H \to 2^G, Y \mapsto f^{-1}Y.$$

Further let φ^- denote the residuated map of φ w.r.t. $(\mathbb{E}, \mathbb{D})$, that is, $(\mathbb{E}, \mathbb{D}, \varphi^-, \varphi)$ is a poset adjunction. Then, obviously, $(\mathbb{D}^{\mathrm{op}}, \mathbb{E}^{\mathrm{op}}, \varphi, \varphi^-)$ is a poset adjunction too.

For pattern structures the following operators are essential:

$$^{\diamond} : 2^G \to D, X \mapsto \inf_{\mathbb{D}} \delta X$$

$$^{\blacklozenge} : D \to 2^G, d \mapsto \{g \in G \mid d \sqsubseteq \delta g\}$$

$$^{\square} : 2^H \to E, Z \mapsto \inf_{\mathbb{E}} \varepsilon Z$$

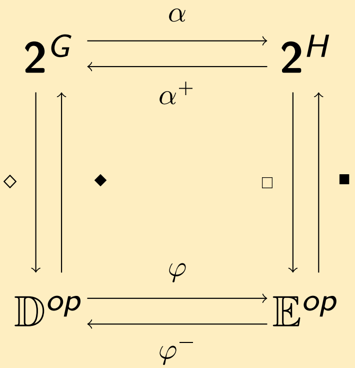
$$^{\blacksquare} : E \to 2^H, e \mapsto \{h \in H \mid e \sqsubseteq \varepsilon h\}$$

It now follows that (α, φ) forms a morphism from the poset adjunction

$$\mathcal{P} = (\mathbf{2}^G, \mathbb{D}^{\mathrm{op}}, ^{\diamond}, ^{\blacklozenge}) \text{ to } \mathcal{Q} = (\mathbf{2}^H, \mathbb{E}^{\mathrm{op}}, ^{\square}, ^{\blacksquare}).$$

In particular, $(fX)^{\square} = \varphi(X^{\diamond})$ holds for all $X \subseteq G$.
We receive the following diagram of adjunctions:

Diagram of Adjunctions



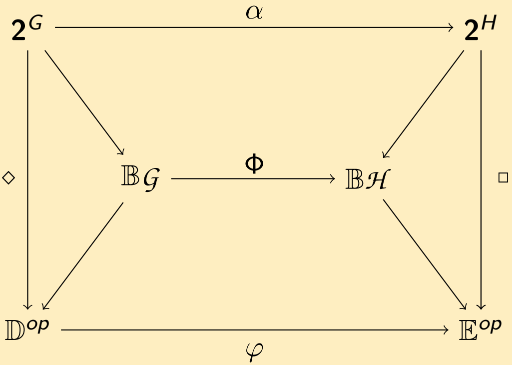
For the following we recall that the concept lattice of $\mathcal{G}$ is given by $\mathbb{B}\mathcal{G} := \mathbb{B}\mathcal{P}$ and the concept lattice of $\mathcal{H}$ is $\mathbb{B}\mathcal{H} := \mathbb{B}\mathcal{Q}$.
Then Theorem 1 yields that the quadruple $(\mathbb{B}\mathcal{G}, \mathbb{B}\mathcal{H}, \Phi, \Phi^{+})$ is an adjunction for

$$\Phi : \mathrm{B}\mathcal{G} \to \mathrm{B}\mathcal{H}, (X, d) \mapsto ((\varphi d)^{\blacksquare}, \varphi d)$$

and

$$\Phi^{+} : \mathrm{B}\mathcal{H} \to \mathrm{B}\mathcal{G}, (Z, e) \mapsto (f^{-1}Z, (f^{-1}Z)^{\diamond}).$$

Diagram with Concept Lattices



Remark

By Theorem 2 the following hold:

(1) If f is surjective then Φ is surjective too.

(2) If φ is injective then Φ is injective too.

(3) If f is surjective and φ is injective then Φ is an isomorphism from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H}$.

Theorem

Let $\mathcal{G} = (G, \mathbb{D}, \delta)$ and $\mathcal{H} = (H, \mathbb{E}, \epsilon)$ be pattern structure. And let $\mathcal{G}^{\bullet} = (G, \mathbb{C}_{\mathcal{G}}, \delta^{\bullet})$ be the pattern structure induced by $\mathcal{G}$ via $\delta^{\bullet} : G \to C_{\mathcal{G}}$, $g \mapsto \delta g$. It follows $\mathbb{B}\mathcal{G}^{\bullet} = \mathbb{B}\mathcal{G}$. Further let (f, φ) be a pattern morphism from $\mathcal{G}^{\bullet}$ to $\mathcal{H}$. Then with the notation introduced in the previous theorem, the map Φ from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H}$ is residuated. If f is surjective then so is Φ, if φ is injective then so is Φ. If f is surjective and φ is injective then Φ is an isomorphism from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H}$.

Definition

The **representation context** of a pattern structure $\mathcal{G} = (G, \mathbb{D}, \delta)$ w.r.t. a subset M of D is given by $\mathbb{K}(\mathcal{G}, M) := (G, M, I)$ with $I := \{(g, m) \in G \times M \mid m \sqsubseteq \delta g\}$.

Theorem

Let $\mathcal{G} = (G, \mathbb{D}, \delta)$ be a pattern structure and let M be a subset of D. The pattern structure associated with the representation context $\mathbb{K}(\mathcal{G}, M)$ is given by $\mathcal{H} := (G, \mathbf{2}^M, \varepsilon)$ with $\varepsilon : G \to 2^M, g \mapsto \downarrow_M \delta g$ where $\downarrow_M d := \{m \in M \mid m \sqsubseteq d\}$ for all $d \in D$. In particular, the concept lattice of $\mathbb{K}(\mathcal{G}, M)$ is given by $\mathbb{B}\mathbb{K}(\mathcal{G}, M) = \mathbb{B}\mathcal{H}$.
Using the notation from the previous theorem, (id_G, φ) is a pattern morphism from $\mathcal{G}^\bullet$ to $\mathcal{H}$ for $\varphi : C_{\mathcal{G}} \to 2^M, x \mapsto \downarrow_M x$. Furthermore, the map Φ from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathcal{H} = \mathbb{B}\mathbb{K}(\mathcal{G}, M)$ is a residuated surjection. In case M is join-dense w.r.t. $\mathbb{C}_{\mathcal{G}}$ (that is, φ is injective), Φ is an isomorphism from $\mathbb{B}\mathcal{G}$ to $\mathbb{B}\mathbb{K}(\mathcal{G}, M)$.

Diagram of the involved maps

$$
\begin{array}{ccc}
 & X \mapsto X & \\
\mathbf{2}^{G} & \xrightarrow[id]{\quad} & \mathbf{2}^{G} \\
{\scriptstyle X \mapsto X^{\diamond}} \downarrow & & \downarrow {\scriptstyle X \mapsto X^{\square}} \\
\mathbb{C}_{\mathcal{G}} & \xrightarrow{\varphi} & \mathbf{2}^{M} \\
 & d \mapsto \downarrow_{M} d &
\end{array}
$$

Diagram of the involved concept lattices