Towards a sequent calculus for formal contexts

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Introduction

- Category theory and FCA
- Hideo Mori - ChuCors
- Bonds and Chu correspondences
- *-autonomism of ChuCors
- model of linear logic
- ???

linear logic

- If you have 1Eur then you can buy caffe.
- If you have 1Eur then you can buy ice-cream.
- $((A \Rightarrow B) \wedge (A \Rightarrow C)) \Leftrightarrow (A \Rightarrow (B \wedge C))$
- If you have 1Eur then you can buy caffe and ice-cream.
- J.Y. Girard introduced multiplicative and additive conjunction ($\otimes$, $\&$) and implication ($\multimap$, $\rightarrow$)
- $(A \multimap B) \otimes (A \multimap C) \Leftrightarrow ((A \otimes A) \multimap (B \otimes C))$
- $H_2 \otimes H_2 \otimes O_2 \multimap H_2O \otimes H_2O$

*-autonomous category

- category endowed with binary operation $\otimes$ (tensor product) forms symmetric and monoidal category
- internal hom $X \multimap Y$ is a special object of all arrows btw X and Y
- $X \otimes Y \cong (X \multimap Y^*)^*$
- closed category $\mathcal{K}(X \otimes Y, Z) \cong \mathcal{K}(X, Y \multimap Z)$ for any objects X, Y, Z of $\mathcal{K}$
- there exist a dualizing object $\perp$ such that $X \multimap \perp \cong X^*$

Bonds

- let $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ for $i \in \{1, 2\}$ be two formal contexts
- $\beta \subseteq B_1 \times A_2$ is a bond between $\mathcal{C}_1$ and $\mathcal{C}_2$
- rows of β are from $\mathrm{Int}(\mathcal{C}_2)$
- columns of β are from $\mathrm{Ext}(\mathcal{C}_1)$

- equivalent definition
- $\mathrm{Ext}(\langle B_1, A_2, \beta \rangle) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ and $\mathrm{Int}(\langle B_1, A_2, \beta \rangle) \subseteq \mathrm{Int}(\mathcal{C}_2)$

- dual bond $\beta \subseteq B_1 \times B_2$
- $\mathrm{Ext}(\langle B_1, B_2, \beta \rangle) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ and $\mathrm{Int}(\langle B_1, B_2, \beta \rangle) \subseteq \mathrm{Ext}(\mathcal{C}_2)$

product of contexts

- let $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ for $i \in \{1, 2\}$
- product $\mathcal{C}_1 \times \mathcal{C}_2 = \langle B_1 \uplus B_2, A_1 \uplus A_2, R_{1\times 2} \rangle$ where the relation $R_{1\times 2}$
- $((i, b), (j, a)) \in R_{1\times 2}$ if and only if $\big((i = j) \Rightarrow (b, a) \in R_i\big)$ for any $(b, a) \in B_i \times A_j$ and $(i, j) \in \{1, 2\} \times \{1, 2\}$
- concept lattice of $\mathcal{C}_1 \times \mathcal{C}_2$ is isomorphic to product of concept lattices of $\mathcal{C}_1$ and $\mathcal{C}_2$

multiplicative and additive conjunction on formal contexts

- any dual bond as multiplicative conjunction
- any value any dual bond contain extents from both input formal contexts

- product of formal contexts as additive conjunction
- any value contain only one concept of input formal contexts
- disjunctive properties

Sequent calculus

- sequent of Gentzen calculus $\Gamma \vdash \Delta$ means: *from the conjunction of all hypothesis of* Γ *follows some formula of* Δ
- as a conjunction of all contexts we use some *n*-ary bond between input contexts
- $\mathcal{C}$ is a *consequence* of contexts $\mathcal{C}_1, \ldots, \mathcal{C}_n$, denoted as $\mathcal{C}_1, \ldots, \mathcal{C}_n \models \mathcal{C}$ if and only if $\mathcal{C}$ is isomorphic to some *n*-ary bond between input formal contexts $\mathcal{C}_1, \ldots, \mathcal{C}_n$.

n-Bonds

- let $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ be formal contexts for $i \in \{1, \dots, n\}$
- $\beta \subseteq \prod_{i=1}^{n} B_i$ is a *dual n-ary bond* between $\{\mathcal{C}_i\}_{i=1}^{n}$
- if for all $i \in \{1, 2, \dots, n\}$ and any $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n) \in \prod_{j=1, j \neq i}^{n} B_i$ it holds that

$$\beta(b_1, \dots, b_{i-1}, -, b_{i+1}, \dots, b_n) \in \mathrm{Ext}(\mathcal{C}_i)$$

Axiom rule

As a unary bond we use a context itself

$$\overline{\mathcal{C} \models \mathcal{C}}$$

Constants rule

Due to isomorphism $\mathrm{Bonds}\,(\mathcal{C},\top) \cong \mathrm{Ext}(\mathcal{C})$ where $\top = \langle\{\diamond\},\{\diamond\},\neq\rangle$ we can write the following rule

$$\frac{\mathcal{C}_1,\ldots,\mathcal{C}_n \models \mathcal{C}}{\top,\mathcal{C}_1,\ldots,\mathcal{C}_n \models \mathcal{C}}$$

$\otimes$-left

From the definition of $\models$ and associativity of tensor product, or of associativity inside of n-ary product, we can write

$$\frac{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n, \models \mathcal{C}}{\mathcal{C}_1, \ldots, \mathcal{C}_{n-2}, (\mathcal{C}_{n-1} \otimes \mathcal{C}_n) \models \mathcal{C}}$$

$\otimes$-right

Any dual bond between n-ary and m-ary bonds is an $n+m$-ary bond between all input formal contexts

$$\frac{\mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{C} \qquad \mathcal{D}_1 \ldots, \mathcal{D}_m \models \mathcal{D}}{\mathcal{C}_1 \ldots, \mathcal{C}_n, \mathcal{D}_1 \ldots, \mathcal{D}_m \models \mathcal{C} \otimes \mathcal{D}}$$

$\times$-left]

Due to the distributivity of tensor and categorical product on formal contexts, *n*-ary bond between $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n \times \mathcal{D}$ is equal to product of *n*-ary bonds between $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n$ and $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{D}$. Hence it is easy to use a full relation as the *n*-ary bond between $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{D}$ to obtain the following rule.

$$\frac{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n, \models \mathcal{C}}{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n \times \mathcal{D} \models \mathcal{C}}$$

$\times$-right

One of the possibilities here is to add to hypothesis a special context, product of two singletons $\top \times \top$.

$$\frac{\mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{D}_1 \qquad \mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{D}_2}{\mathcal{C}_1 \ldots, \mathcal{C}_n \models \mathcal{D}_1 \times \mathcal{D}_2}$$

Thank you for your attention!