Relational proportions between objects and attributes

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Analogical proportion

- analogical proportions:
  statements of the form "A is to B as C is to D"
- when A, B, C, D are represented
  in terms of the **same** features,
  in the setting of *Boolean logic*:

  "A differs from B as C differs from D
  and B differs from A as D differs from C"

  "the veal is to the cow as the lamb is to the sheep"
- extended using *multiple-valued logic* for handling numerical features

Relational proportion

- Statement "Carlsen is to chess as Mozart is to music" relates **2** types of items, here people and activities
- a *special case* of analogical proportion
- "object A has the same relationship with attribute a as object B with attribute b"
- the nature of relational proportions suggests to handle them in the setting of *formal concept analysis*
- defining analogical proportions

  between formal concepts

Contents

Analogical proportions

Definition (1)

An analogical proportion (AP) on a set X is a quaternary relation on X, *i.e.* a subset of X^4. An element of this subset, written $(x : y :: z : t)$, which reads 'x is to y as z is to t', must obey the following axioms:

1. *Reflexivity of 'as'*: $(x : y :: x : y)$
2. *Symmetry of 'as'*: $(x : y :: z : t) \Leftrightarrow (z : t :: x : y)$
3. *Central permut.*: $(x : y :: z : t) \Leftrightarrow (x : z :: y : t)$

8 equivalent forms : $(x : y :: z : t)$, $(z : t :: x : y)$, $(y : x :: t : z)$, $(t : z :: y : x)$, $(z : x :: t : y)$, $(t : y :: z : x)$, $(x : z :: y : t)$ and $(y : t :: x : z)$

Analogical proportions in lattices (ECAI-2014)

Definition (2)

A 4-tuple (x, y, z, t) of a lattice $(L, \vee, \wedge, \leq)^4$ is a Factorial Analogical Proportion (**FAP**) $(x : y :: z : t)$ iff:

$$
\begin{aligned}
x &= (x \wedge y) \vee (x \wedge z) & \quad x &= (x \vee y) \wedge (x \vee z)\\
y &= (x \wedge y) \vee (y \wedge t) & \quad y &= (x \vee y) \wedge (y \vee t)\\
z &= (z \wedge t) \vee (x \wedge z) & \quad z &= (z \vee t) \wedge (x \vee z)\\
t &= (z \wedge t) \vee (y \wedge t) & \quad t &= (z \vee t) \wedge (y \vee t)
\end{aligned}
$$

Definition (3)

A 4-tuple (x, y, z, t) of $(L, \vee, \wedge, \leq)^4$ is a Weak Analogical Proportion (**WAP**) when $x \wedge t = y \wedge z$ and $x \vee t = y \vee z$. It is denoted $x : y \ WAP \ z : t$.

Example of FAP

- a FAP is a WAP and the converse is false

- **Proposition (1)**

  Let y and z be two elements of a lattice, the proportion

$$y : y \vee z :: y \wedge z : z$$

  is a FAP.

Formal concept analysis

- a set $\mathcal{O}$ of objects a set $\mathcal{A}$ of attributes
  The tuple $(\mathcal{O}, \mathcal{A}, R)$ is called a *formal context*.
- $(o, a) \in R$ or oRa means object o has attribute a.
  $o^{\uparrow} = \{a \in \mathcal{A} | (o, a) \in R\}$ the attribute set of object o
  $a^{\downarrow} = \{o \in \mathcal{O} | (o, a) \in R\}$ object set having attribute a.
- for any subset $\mathbf{o}$ of objects, $\mathbf{o}^{\uparrow} = \{a \in \mathcal{A} | a^{\downarrow} \supseteq \mathbf{o}\}$
  for any subset $\mathbf{a}$ of attribures, $\mathbf{a}^{\downarrow} = \{o \in \mathcal{O} | o^{\uparrow} \supseteq \mathbf{a}\}$
- Then a *formal concept* is defined as a pair $(\mathbf{o}, \mathbf{a})$, such that $\mathbf{a}^{\downarrow} = \mathbf{o}$ and $\mathbf{o}^{\uparrow} = \mathbf{a}$. $\mathbf{o}$ is the *extension* of the concept and $\mathbf{a}$ its *intension*.
- The set of all formal concepts is equipped with a partial order $\leq$: $(\mathbf{o}_1, \mathbf{a}_1) \leq (\mathbf{o}_2, \mathbf{a}_2)$ iff $\mathbf{o}_1 \subseteq \mathbf{o}_2$ (or, equivalently, $\mathbf{a}_2 \subseteq \mathbf{a}_1$) $\Rightarrow$ the *concept lattice* of R

Preliminaries

- 1. Given two concepts $x = (\mathbf{o}_x, \mathbf{a}_x)$ and $y = (\mathbf{o}_y, \mathbf{a}_y)$, one has $(\mathbf{o}_x \cup \mathbf{o}_y)^{\uparrow} = \mathbf{a}_x \cap \mathbf{a}_y$ and $(\mathbf{a}_x \cup \mathbf{a}_y)^{\downarrow} = \mathbf{o}_x \cap \mathbf{o}_y$.
- 2. Given two concepts $x = (\mathbf{o}_x, \mathbf{a}_x)$ and $y = (\mathbf{o}_y, \mathbf{a}_y)$, one has $\mathbf{o}_x \cup \mathbf{o}_y \subseteq \mathbf{o}_{x \vee y}$, $\mathbf{o}_x \cap \mathbf{o}_y = \mathbf{o}_{x \wedge y}$, $\mathbf{a}_x \cup \mathbf{a}_y \subseteq \mathbf{a}_{x \wedge y}$ and $\mathbf{a}_x \cap \mathbf{a}_y = \mathbf{a}_{x \vee y}$.
- 3. Let $\mathbf{o}$ (resp. $\mathbf{a}$) be a subset of $\mathcal{O}$ (resp. $\mathcal{A}$), there exists at most one concept x such that $\mathbf{o}_x = \mathbf{o}$ (resp. $\mathbf{a}_x = \mathbf{a}$).

WAP proportions and FCA

- **Proposition (2)**

  Let x, y, z and t be four concepts, one has:
  $(x \vee t = y \vee z \text{ iff } \mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z)$
  and $(x \wedge t = y \wedge z \text{ iff } \mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z)$.
  As consequence, $(x : y \text{ WAP } z : t)$
  iff $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ *and* $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$.

- **Proposition (3)**

  Let x, y, z and t be four concepts, if $(\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$ *or* $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t)$ *then* $x : y \text{ WAP } z : t$.

The converse is false

FAP proportions and FCA

Proposition (4)

Let x, y, z and t be four concepts, if ($\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z$ and $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z$) then the FAP $x : y :: z : t$ holds.

Corollary *Let x, y, z and t be four concepts, the following two conjunctions are equivalent:*

$$\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z \quad \textit{and} \quad \mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z$$

$$\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t \quad \textit{and} \quad \mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$$

A particular case of FAP between concepts called Strong Analogical Proportion: $x : y \, SAP \, z : t$, where 4 concepts are in analogical proportion on attributes and on objects.

If $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$ and $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$ then $x : y \; FAP \; z : t$. the reciprocal is false

From a RP to concepts in AP -1

"Massimiliano Alajmo is the Mozart of Italian cooking"
background knowledge : music and Italian cooking are disciplines practiced by humans,
with different levels of ability,
Mozart is a musician and is a genius in music discipline.
Since everybody is not "a genius", there exist many "ordinary gifted" musicians. leading to the following formal context:

	a_1	a_2	a_3
o_1	×	×	
o_2	×		×

where o_1 stands for Mozart, o_2 for one of "ordinary gifted" musicians, a_1 is the attribute "practices music", a_2 "is a genius" and a_3 "has an ordinary ability".

From a RP to concepts in AP - 2

"Massimiliano Alajmo is the Mozart of Italian cooking"

	a_1	a_2	a_3	a_4
o_1	×	×		
o_2	×		×	
o_3		×		×
o_4			×	×

where o_3 stands for Alajmo, o_4 an ordinary gifted Italian cook and a_4 Italian cooking. This context is called the *analogical context*. Considering the associated concept lattice, the closest analogical proportion to "Alajmo is the Mozart of Italian cooking" is $(\{o_3\}, \{a_2, a_4\})$: $(\{o_4\}, \{a_3, a_4\})$ *WAP* $(\{o_1\}, \{a_1, a_2\})$: $(\{o_2\}, \{a_1, a_3\})$ which translates into "Mozart is to some ordinary musician as Alajmo is to some ordinary cook".

Analogical complex

It is a subcontext of a formal context described by:

			×	×
		×		×
	×		×	
	×	×		

associated with matrix $AS = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

Definition (4)

Given $(\mathcal{O}, \mathcal{A}, R)$, a set of objects $\mathbf{o} \subseteq \mathcal{O}$, $\mathbf{o} = \mathbf{o}_1 \cup \mathbf{o}_2 \cup \mathbf{o}_3 \cup \mathbf{o}_4$, a set of attributes $\mathbf{a} \subseteq \mathcal{A}$, $\mathbf{a} = \mathbf{a}_1 \cup \mathbf{a}_2 \cup \mathbf{a}_3 \cup \mathbf{a}_4$, and a binary relation R, the subcontext $(\mathbf{o}, \mathbf{a})$ forms an analogical complex $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ iff

1. the binary relation is compatible with the analogical schema AS:
$\forall i \in [1,4],\ \forall o \in \mathbf{o}_i,\ \forall j \in [1,4],\ \forall a \in \mathbf{a}_j,\ ((o,a) \in R) \Leftrightarrow AS(i,j).$
2. The context is *maximal* with respect to the first property ($\oplus$ denotes the exclusive or and $\setminus$ the set-theoretic difference):
$\forall o \in \mathcal{O} \setminus \mathbf{o}, \forall i \in [1,4],\ \exists j \in [1,4], \exists a \in \mathbf{a}_j, ((o,a) \in R) \oplus AS(i,j).$
$\forall a \in \mathcal{A} \setminus \mathbf{a}, \forall j \in [1,4],\ \exists i \in [1,4], \exists o \in \mathbf{o}_i, ((o,a) \in R) \oplus AS(i,j).$

An analogical complex is complete if none of $\mathbf{a}_1, \ldots, \mathbf{a}_4, \mathbf{o}_1, \ldots, \mathbf{o}_4$ is empty

Example

Let us consider a subcontext, called SmallZoo, extracted from Zoo data base, it has been shown that 24 analogical complexes (18 complete ones) can be derived, like the following complete one:

	SmallZoo	hair a_0	feathers a_1	eggs a_2	milk a_3	airborne a_4	aquatic a_5	predator a_6	toothed a_7
o_0	aardvark	×			×			×	×
o_1	chicken		×	×		×			
o_2	crow		×	×		×		×	
o_3	dolphin				×		×	×	×
o_4	duck		×	×		×	×		
o_5	fruitbat	×			×	×			×
o_6	kiwi		×	×				×	
o_7	mink	×			×		×	×	×
o_8	penguin		×	×			×	×	
o_9	platypus	×		×	×		×	×	

		$\mathbf{a}_1$	$\mathbf{a}_2$			$\mathbf{a}_3$		$\mathbf{a}_4$
		a_5	a_0	a_3	a_7	a_1	a_2	a_4
$\mathbf{o}_1$	o_1					×	×	×
	o_2					×	×	×
$\mathbf{o}_2$	o_5		×	×	×			×
$\mathbf{o}_3$	o_8	×				×	×	
$\mathbf{o}_4$	o_7	×	×	×	×			

Definition (5)

Let $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ be a *complete analogical complex* in a formal context, the following sets of objects and attributes are said to be in the *formal relational proportion* ($\mathbf{o}_1$ is to $\mathbf{a}_3$ as $\mathbf{o}_2$ is to $\mathbf{a}_2$), and we write: $(\mathbf{o}_1 \updownarrow \mathbf{a}_3 \Updownarrow \mathbf{o}_2 \updownarrow \mathbf{a}_2)$.

WAP and analogical complex

Definition (6)

Let us consider $(x : y \; WAP \; z : t)$, this WAP is complete when

1. either $(\mathbf{a}_x \cap \mathbf{a}_y) \setminus \mathbf{a}_\cap$, $(\mathbf{a}_x \cap \mathbf{a}_z) \setminus \mathbf{a}_\cap$, $(\mathbf{a}_y \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap$ and $(\mathbf{a}_z \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap$ are *nonempty* (called complete WAP through attributes),
2. or $(\mathbf{o}_x \cap \mathbf{o}_y) \setminus \mathbf{o}_\cap$, $(\mathbf{o}_x \cap \mathbf{o}_z) \setminus \mathbf{o}_\cap$, $(\mathbf{o}_y \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap$ and $(\mathbf{o}_z \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap$ are *nonempty* (called complete WAP through objects).

where $\mathbf{a}_\cap = \mathbf{a}_x \cap \mathbf{a}_y \cap \mathbf{a}_z \cap \mathbf{a}_t$ and $\mathbf{o}_\cap = \mathbf{o}_x \cap \mathbf{o}_y \cap \mathbf{o}_z \cap \mathbf{o}_t$.

Proposition (5)

1. *A complete WAP is an antichain of concepts.*
2. *For a complete WAP through attributes,* $(x \vee y)$, $(x \vee z)$, $(y \vee t)$ *and* $(z \vee t)$ *are in antichain. Similarly, for a complete WAP through objects,* $(x \wedge y)$, $(x \wedge z)$, $(y \wedge t)$ *and* $(z \wedge t)$ *are in antichain.*
3. *A FAP in antichain forms a complete WAP through attributes and objects, and reciprocally.*

Example

In SmallZoo, $x = (\{o_1, o_2, o_4\}, \{a_1, a_2, a_4\})$, $y = (\{o_5\}, \{a_0, a_3, a_4, a_7\})$, $z = (\{o_4, o_8\}, \{a_1, a_2, a_5\})$, $t = (\{o_7, o_9\}, \{a_0, a_3, a_5, a_6\})$ are concepts in complete WAP through attributes. At the beginning, $\mathbf{o}_1 = \{o_1, o_2\}$, $\mathbf{o}_2 = \{o_5\}$, $\mathbf{o}_3 = \{o_8\}$, $\mathbf{o}_4 = \{o_7, o_9\}$, $\mathbf{a}_1 = \{a_5\}$, $\mathbf{a}_2 = \{a_0, a_3\}$, $\mathbf{a}_3 = \{a_1, a_2\}$ and $\mathbf{a}_4 = \{a_4\}$ the first postprocessing can remove (either o_9 or) a_2:

	a_5	a_0	a_3	a_1	a_2	a_4
o_1				×	×	×
o_2				×	×	×
o_5		×	×			×
o_8	×			×	×	
o_7	×	×	×			
o_9	×	×	×		×	

		$\mathbf{a}_1$	$\mathbf{a}_2$		$\mathbf{a}_3$	$\mathbf{a}_4$
		a_5	a_0	a_3	a_1	a_4
$\mathbf{o}_1$	o_1				×	×
	o_2				×	×
$\mathbf{o}_2$	o_5		×	×		×
$\mathbf{o}_3$	o_8	×			×	
$\mathbf{o}_4$	o_7	×	×	×		
	o_9	×	×	×		

Concluding remarks

- two cognitive capabilities, *conceptual categorization* and *analogical reasoning* can be handled together in the setting of formal concept analysis
- *relational proportions* offer a basis
  for concise forms of explanations
  "object A is to attribute a as object B is to attribute b"
  provides an argument for stating that
  "object A is the B of a"
  when A possesses the same well-known features
- connection a recent proposal based on *antichains* (M. Ojeda-Aciego et al.)
- bridging the gap with *computational linguistics* works