Rectangle and Square Coverings of Tolerance Spaces and their Direct Product

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FCA4AI, 13.07.2018, Stockholm, Sweden

Tolerance Spaces

- *tolerance relation:* reflexive and symmetric relation $\tau \subseteq V \times V$
- *tolerance space:* $\mathbb{T} := (V, \tau)$
- formal context (V, V, τ)
- maximal rectangles and squares:

$I:$	m_1	m_2	m_3	m_4
g_1	0	0	1	0
g_2	0	0	0	1
g_3	1	0	0	1
g_4	0	1	1	1

$\tau:$	a	b	c	d
a	1	1	1	0
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- set of all *maximal squares:* $\mathrm{Sq}(\mathbb{T}) \subseteq \mathfrak{B}(\mathbb{T})$

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Square and Rectangle Cover Number

- $I = \bigcup\{A \times B \mid (A, B) \in \mathfrak{B}(\mathbb{K})\}$
- $\tau = \bigcup\{S \mid S \in \mathrm{Sq}(\mathbb{T})\}$
- the *rectangle cover* number of $\mathbb{K}$:

$$\mathrm{rc}(\mathbb{K}) := \min\{\#\mathcal{F} \mid \mathcal{F} \subseteq \mathfrak{B}(\mathbb{T}),\ I = \bigcup \mathcal{F}\}$$

- the *square cover* number of $\mathbb{T}$:

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Definitions and Facts I

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- *complementary context*:
  $\mathbb{K}^c = (G, M, I^c) := (G, M, (G \times M) - I)$
- *crossed* and *co-crossed* contexts:

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Definitions and Facts II

- *direct sum* $\mathbb{K}_1 \oplus \mathbb{K}_2 := (G_1 \dot\cup G_2, M_1 \dot\cup M_2, I_1 \oplus I_2)$

$I_1 \oplus I_2$:	M_1	M_2
G_1	I_1	$G_1 \times M_2$
G_2	$G_2 \times M_1$	I_2

- $\underline{\mathfrak{B}}(\mathbb{K}_1 \oplus \mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1) \times \underline{\mathfrak{B}}(\mathbb{K}_2)$

Definitions and Facts III

- *direct product*: $\mathbb{K}_1 \check{\times} \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, I_1 \check{\times} I_2)$

  $$((g,h),(m,n)) \in I_1 \check{\times} I_2 \;:\Longleftrightarrow\; (g,m) \in I_1 \text{ or } (h,n) \in I_2$$

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Rectangle Cover Number

- it holds that:

$$I_1 \check{\times} I_2 = (G_1 \times M_1) \hat{\times} I_2 \cup I_1 \hat{\times} (G_2 \times M_2)$$

- it follows that:

$$\mathrm{rc}(\mathbb{K}_1 \check{\times} \mathbb{K}_2) \leq \mathrm{rc}(\mathbb{K}_1) + \mathrm{rc}(\mathbb{K}_2)$$

cover problem $\iff$ intersection prob. $\iff$ lattice dimension

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Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be co-crossed contexts. For the rectangle cover number of their direct product it holds that:

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Upper Bounds

- $\mathrm{rc}(\mathbb{K}) \leq \min(|G|, |M|) \Longrightarrow \mathrm{rc}(\mathbb{T}) \leq |V|$
- $\mathrm{rc}(\mathbb{T}) \leq \mathrm{sc}(\mathbb{T})$
- for $|V| = 1, 2, 3, 4 : \quad \mathrm{sc}(\mathbb{T}) \leq |V|$
- for $|V| \geq 4 : \quad \mathrm{sc}(\mathbb{T}) \leq \lfloor |V|^2/4 \rfloor$

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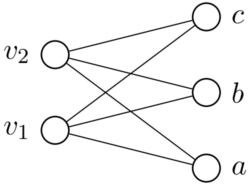
Example

- $6 = \mathrm{sc}(K_{2,3}^{\mathrm{ref}}) > \mathrm{rc}(K_{2,3}^{\mathrm{ref}}) = 5$



	v_1	v_2	a	b	c
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Balanced Covering Property

Definition

We say that a tolerance space $\mathbb{T}$ has the *balanced covering property* (in short *BCP*) if $\mathrm{sc}(\mathbb{T}) = \mathrm{rc}(\mathbb{T})$.

computational experiments:

- non-isomorphic tolerance spaces with $|V| \leq 10$: 12.293.433
- tolerance spaces with $|V| \leq 10$ and BCP: 2.553.962

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Tolerance Spaces induced by irredundant Coverings

τ :	a	b	c	d	e
a	1	1	0	0	0
b	1	1	1	1	0
c	0	1	1	1	0
d	0	1	1	1	1
e	0	0	0	1	1

Tolerance Spaces with the BCP

$\tau:$	a	b	c	d
a	1	0	1	1
b	0	1	1	1
c	1	1	1	0
d	1	1	0	1

Tolerance Spaces with the BCP

- $\mathbb{T} := (\mathbb{K} \dot{\cup} \mathbb{K}^d)^{\text{ref}}$ with $\mathbb{K} = (G, M, I)$

$(I \dot{\cup} I^{-1})^{\text{ref}}$:	G	M
G	E_G	I
M	I^{-1}	E_M

- $\{a\}, A \subseteq G$ and $\{b\}, B \subseteq M$
- $(\{a\}, \{a\} \cup A^I), (\{b\}, B_I \cup \{b\}), (\{a\} \cup B, \{a\}), (A \cup \{b\}, \{b\})$
- (A, A^I), (B, B_I) and $(\{a\} \cup \{b\}, \{a\} \cup \{b\})$
- $|G| + |M| < |I| \Rightarrow \text{rc}(\mathbb{T}) = |G| + |M| < \text{sc}(\mathbb{T}) = |I|$
- $|G| + |M| \geq |I| \Rightarrow \text{rc}(\mathbb{T}) = \text{sc}(\mathbb{T}) \leq |G| + |M|$

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- $\mathbb{T} := (\mathbb{K} \dot\cup \mathbb{K}^d)^{\text{ref}}$ with $\mathbb{K} = (G, M, I)$

$(I \dot\cup I^{-1})^{\text{ref}}$:	G	M
G	E_G	I
M	I^{-1}	E_M

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- sufficient condition for additivity of the rectangle cover number with respect to the direct product of formal contexts
- square cover number and rectangle cover number of tolerance spaces
- example classes for tolerance spaces with the balanced covering property
- sufficient condition for additivity of the square cover number with respect to the direct product of tolerance spaces

Thank you for your attention!

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