Generalized metrics with applications to ratings and formal concept analysis

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July 13, 2018

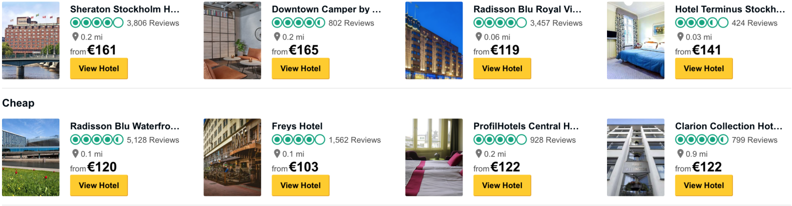
Credit Rating Scales: Examples

Moody's	Standard & Poor's	Fitch	AM Best	Credit worthiness
Aaa	AAA	AAA	aaa	An obligor has EXTREMELY STRONG capacity to meet its financial commitments.
Aa1	AA+	AA+	aa+	An obligor has VERY STRONG capacity to meet its financial commitments. It differs from the highest rated obligors only in small degree.
Aa2	AA	AA	aa	
Aa3	AA-	AA-	aa-	
A1	A+	A+	a+	An obligor has STRONG capacity to meet its financial commitments but is somewhat more susceptible to the adverse effects of changes in circumstances and economic conditions than obligors in higher-rated categories.
A2	A	A	a	
A3	A-	A-	a-	
Baa1	BBB+	BBB+	bbb+	An obligor has ADEQUATE capacity to meet its financial commitments. However, adverse economic conditions or changing circumstances are more likely to lead to a weakened capacity of the obligor to meet its financial commitments.
Baa2	BBB	BBB	bbb	
Baa3	BBB-	BBB-	bbb-	
Ba1	BB+	BB+	bb+	An obligor is LESS VULNERABLE in the near term than other lower-rated obligors. However, it faces major ongoing uncertainties and exposure to adverse business, financial, or economic conditions which could lead to the obligor's inadequate capacity to meet its financial commitments.
Ba2	BB	BB	bb	
Ba3	BB-	BB-	bb-	
B1	B+	B+	b+	An obligor is MORE VULNERABLE than the obligors rated 'BB', but the obligor currently has the capacity to meet its financial commitments. Adverse business, financial, or economic conditions will likely impair the obligor's capacity or willingness to meet its financial commitments.
B2	B	B	b	
B3	B-	B-	b-	
Caa	CCC	CCC	ccc	An obligor is CURRENTLY VULNERABLE, and is dependent upon favourable business, financial, and economic conditions to meet its financial commitments.
Ca	CC	CC	cc	An obligor is CURRENTLY HIGHLY-VULNERABLE.
	C	C	c	The obligor is CURRENTLY HIGHLY-VULNERABLE to nonpayment. May be used where a bankruptcy petition has been filed.
C	D	D	d	An obligor has failed to pay one or more of its financial obligations (rated or unrated) when it became due.
e, p	pr	Expected		Preliminary ratings may be assigned to obligations pending receipt of final documentation and legal opinions. The final rating may differ from the preliminary rating.
WR				Rating withdrawn for reasons including: debt maturity, calls, puts, conversions, etc., or business reasons (e.g. change in the size of a debt issue), or the issuer defaults.
unsolicited	unsolicited			This rating was initiated by the ratings agency and not requested by the issuer.
	SD	RD		This rating is assigned when the agency believes that the obligor has selectively defaulted on a specific issue or class of obligations but it will continue to meet its payment obligations on other

— Investment grade —

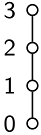
— "Junk" or sub-investment grade —

Grading: Example



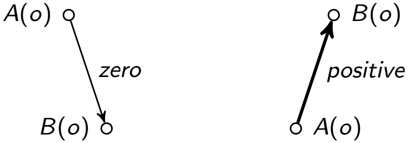
Rating: What is it?

- O set of *objects* to be rated
  - prominent examples are financial entities which issue debt
- There are different (credit) rating agencies applying different ratings, where a (credit) *rating* is a mapping $A : O \to S$ to a rating scale S
- *Rating scale* is a finite chain $S = C(n) := \{0, \ldots, n\}$
  - S naturally and totally ordered by "$\leq$"
  - n:= *length* of chain S
  - "0" represents the lowest credit quality, "n" the highest



Motivation for Directed Distances

- **Question:** Given two ratings A and B from different sources
  - Which one is more *progressive?*
- "Bicycle Distance":



"Only counting when rating $B(o)$ is higher than $A(o)$"

Measure of Progressivity: a Directed Distance

Input: O a set, a finite chain $S = \{0, \dots, n\}$, two ratings $A, B : O \to S$

Definition (Progressivity of rating B given rating A)

$$D^+(A, B) := \sum_{o \in O\colon A(o) \leq B(o)} \mathit{rank}\ B(o) - \mathit{rank}\ A(o)$$

- "natural rank function in a chain": $\mathit{rank}\ s := s$, $s \in S$
- $D^+(A, B)$ "well-defined" and finite if O is finite
- $D^+(A, B) \geq 0$
- $D^+(A, B) = 0$ iff $\forall o \in O : B(o) \leq A(o)$
- $D^+(A, B)$ "triangular", i.e. $\forall C\colon O \to S$: $D^+(A, B) \ \leq \ D^+(A, C) \ + \ D^+(C, B)$: corollary of a theorem below

Measure of Conservatism: the same Directed Distance

Input: O a set, a finite chain $S = \{0, \ldots, n\}$, two ratings $A, B : O \to S$

Definition (Conservatism of rating A given rating B)

$$D^-(B, A) := \sum_{o \in O:\ A(o) \leq B(o)} \mathit{rank}\ B(o) - \mathit{rank}\ A(o)$$

- $D^+(A, B) = D^-(B, A)$
- Usually $D^+(A, B) \neq D^+(B, A)$
- If $D^+(A, B) > D^+(B, A)$ then A is more conservative than B, and B is more progressive than A.

The Symmetric Case: Measuring Distance between Ratings

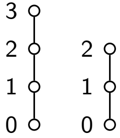
Input: O a finite set, a finite chain $S = \{0, \ldots, n\}$, two ratings $A, B : O \to S$

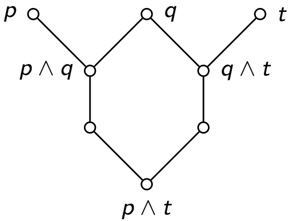
Definition (Distance between ratings A and B)

$$D(A, B) := D^+(A, B) + D^+(B, A)$$

- "symmetric": $D(A, B) = D(B, A)$
- $D(A, B) = 0 = D(B, A)$ iff $A = B$

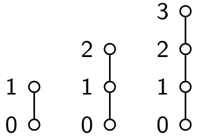
Questions and Generalizations





- "Scaling": Scales do not need to be identical. Different raters use different scales. What can we do about it?
- Why using linear orders at all? What about posets as target of ratings? Which posets will work?

Algorithm for Scaling



- Embedding (order preserving injection) one chain into the other
- Embedding $C(1)$ into $C(2)$: 3 possibilities
- Embedding $C(1)$ into $C(3)$: 6 possibilities

Algorithm for Scaling

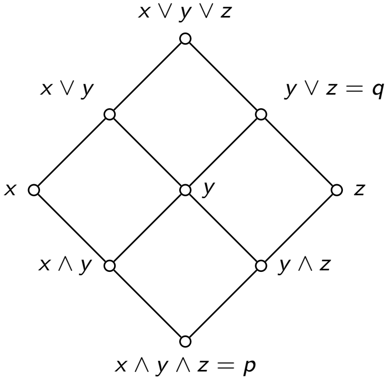
Input: O a finite set, ratings $A : O \to C(k)$, $B : O \to C(n)$, $k \leq n$

Algorithm (Scaling with minimal distance)

- *Run through all embeddings* $E_i : C(k) \to C(n)$
- *Calculate* $E_i \circ A$ *and* $D(B,\ E_i \circ A)$ *for each embedding* E_i
- *Pick (one of) the* E_i *with minimal distance* $D(B,\ E_i \circ A)$

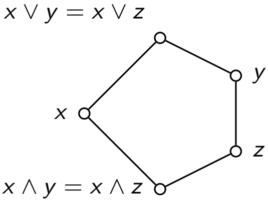
- Works well if the number of embeddings $\binom{n+1}{k+1}$ is not too high.
- Develop smarter matching algorithm (and get rich)

Directed Distances for Ratings with Target Posets other than Chains I



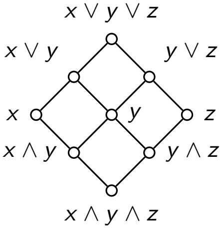
- If $p \leq q$, then p and q are contained in a chain, we can try to use our distance above which compares positions in a chain

Directed Distances for Ratings with Target Posets other than Chains II



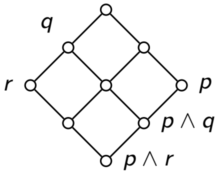
- But we need a condition, which makes (length of) *maximal chains* unique.
- A chain C is called *maximal* if, for any chain D, $C \subseteq D$ implies $C = D$.
- Such a condition is the famous *Jordan-Dedekind chain condition*.

Directed Distances for Ratings with Target Posets other than Chains III



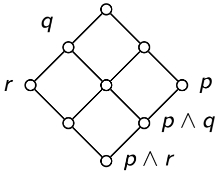
- A poset is said to satisfy the *Jordan-Dedekind chain condition* if any two maximal chains between the same elements have the same finite length.

Directed Distances for Ratings with Target Posets other than Chains IV



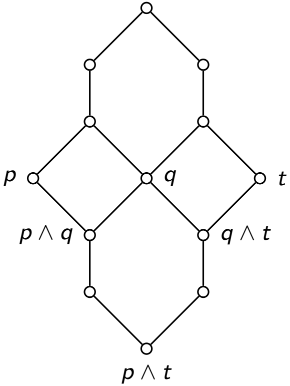
- There is neither a chain containing (p, q) nor (p, r).
- Agreement $p \wedge q$ higher than agreement $p \wedge r$.
- Take one maximal chain between $(p \wedge q, p)$, resp. between $(p \wedge r, p)$.
- Compare rank p with rank $(p \wedge q)$ resp. with rank $(p \wedge r)$.

Directed Distances for Ratings with Target Posets other than Chains V



- $d(p, q) :=$ rank $p -$ rank $p \wedge q$
- $d(p, q) = 1$, $d(p, r) = 2$, $d(q, r) = 1$ and $d(r, q) = 0$
- Do we always get $d(p, q) \leq d(p, r) + d(r, q)$?
- Gives this construction always a "triangular" d?

WARNING: Example of a "Non-Triangular-Metric"



$$d(p,t) = rp - r(p \wedge t) = 3 - 0 = 3$$
$$d(p,q) = rp - r(p \wedge q) = 3 - 2 = 1$$
$$d(q,t) = rq - r(q \wedge t) = 3 - 2 = 1$$

Hence,

$$d(p,t) = 3 > 2 = d(p,q) + d(q,t)$$

Thus, d is <u>not</u> a triangular metric!

Generalized Framework: Functorial Maps

Input: $\mathbb{P} = (P, \leq_{\mathbb{P}})$ a poset (i.e. $\leq_{\mathbb{P}} \subseteq P \times P$) &
$\mathcal{M} = (M, *, \varepsilon, \leq)$ an ordered monoid

Definition (Functorial map)

A map $\Delta\colon \ \leq_{\mathbb{P}} \ \longrightarrow M$ is called **functorial w. r. t.** $(\mathbb{P}, \mathcal{M})$ if

- for all $p \in P$: $\boxed{\Delta(p,p) = \varepsilon}$
- for all $p, t, q \in P$ with
  $p \leq_{\mathbb{P}} t \leq_{\mathbb{P}} q$: $\boxed{\Delta(p,t) * \Delta(t,q) = \Delta(p,q)}$

Definition (Weakly positive map)

Δ is called **weakly positive** if $\varepsilon \leq \Delta(p,q)$ for all $(p,q) \in \ \leq_{\mathbb{P}}$.

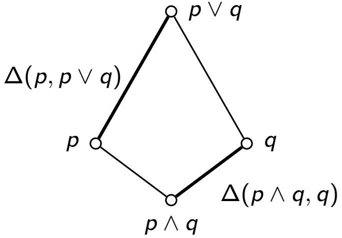
Generalized Framework: Supermodular Maps

Definition (Supermodular maps)

- A map $\Delta\colon \leq_{\mathbb{P}} \longrightarrow M$ is called **supermodular** if

$$\Delta(p \wedge q, q) \leq \Delta(p, p \vee q)$$

holds for all (p, q) s.t. both $p \wedge q$ and $p \vee q$ exist.



Generalized Framework: Algebraic Modeling of Directed Distances

Input: P a set & $\mathcal{M} = (M, *, \varepsilon, \leq)$ an ordered monoid

Definition (Generalized Quasi-Metric: "GQM")

Map $d\colon P \times P \longrightarrow M$ is a GQM w. r. t. $(P, \mathcal{M})$ if it is

- "weakly positive": $\boxed{\varepsilon \leq d(p,q)}$ for all $p, q \in P$
- "neutral": $\boxed{d(p,p) = \varepsilon}$ for all $p \in P$
- "triangular": $\boxed{d(p,q) \leq d(p,t) * d(t,q)}$ for all $p, t, q \in P$

Generalized Framework: Question

For a given $\Delta\colon \ \leq_{\mathbb{P}} \longrightarrow M$, does there exist a generalized quasi-metric $d\colon P \times P \longrightarrow M$ w. r. t. $(P, \mathcal{M})$ which *extends* Δ such that $d|_{\leq_{\mathbb{P}}} = \Delta$?

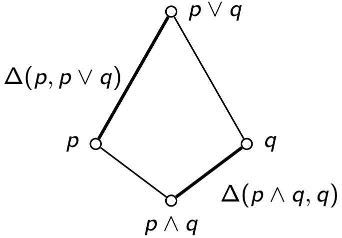
Functorial maps and their role in constructing GQM

Theorem (Supermodular case)

Let $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}})$ be a lattice. If a map $\Delta\colon \leq_{\mathbb{P}} \longrightarrow M$ is weakly positive, supermodular and functorial w. r. t. $(\mathbb{P}, \mathcal{M})$, then

$$\boxed{d\colon P \times P \longrightarrow M, \quad (p, q) \mapsto \Delta(p \wedge q, q)}$$

is a GQM w. r. t. $(\mathbb{P}, \mathcal{M})$.



Conclusion

- In order to compare ratings, we propose a sound directed metric in order to measure how **progressive** or **conservative** ratings are.
- **Scaling:** For chains S, S' of different size we propose an algorithmic solution, which works well if the difference of the length of the two chains is not too big.
- **Getting rich:** Understand existing matching algorithms. Develop smarter ones. Use for apps.
- **Posets as target:** As target other then simply chains there is the huge class of lattices which allow for a (finite) Jordan-Dedekind chain condition together with a supermodular rank function. In particular, distributive (and modular) lattices of finite length will work very well.

Generalized Metrics for Actuaries: Appendix. Proof of Main Theorem

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Algebraic Modeling of Directed Distances

Input: P a set & $\mathcal{M} = (M, *, \varepsilon, \leq)$ an ordered monoid

Definition (Generalized Quasi-Metric: "GQM")

Map $d\colon P \times P \longrightarrow M$ is a GQM w. r. t. $(P, \mathcal{M})$ if it is

- "weakly positive": $\boxed{\varepsilon \leq d(p,q)}$ for all $p, q \in P$
- "neutral": $\boxed{d(p,p) = \varepsilon}$ for all $p \in P$
- "triangular": $\boxed{d(p,q) \leq d(p,t) * d(t,q)}$ for all $p, t, q \in P$

Functorial Maps and their Role in constructing GQM

Recall $d(p, q) := \Delta(p \wedge q, q)$

d: "Weakly positive" and "neutral" clear.

Steps to show that d is "triangular":

1. $t \leq_{\mathbb{P}} x \leq_{\mathbb{P}} z \quad \Longrightarrow \quad \Delta(x, y) \leq \Delta(t, z)$
2. $x \leq_{\mathbb{P}} y \quad \Longrightarrow \quad \Delta(x \wedge y, y \wedge z) \leq \Delta(x, y)$
3. $\Delta(p \wedge q, q) \leq \Delta(p \wedge t, t) * \Delta(t \wedge q, q)$

Functorial maps and their role in constructing GQM

Claim 1 (Interval Property):
$t \leq_{\mathbb{P}} x \leq_{\mathbb{P}} z \quad \Longrightarrow \quad \Delta(x, y) \leq \Delta(t, z)$

Proof. Since Δ is functorial, we obtain

$$\Delta(t, z) = \Delta(t, x) * \Delta(x, y) * \Delta(y, z)$$

As Δ is weakly positive, we get

$$\begin{aligned} \Delta(t, z) &\geq \varepsilon * \Delta(x, y) * \varepsilon \\ &= \Delta(x, y) \end{aligned}$$

Functorial Maps and their Role in constructing GQM

Claim 2 (Meet Property):
$x \leq_{\mathbb{P}} y \quad \implies \quad \Delta(x \wedge y, y \wedge z) \leq \Delta(x, y)$

Proof.

$$\Delta(x \wedge y, y \wedge z) = \Delta(x \wedge (y \wedge z), y \wedge z)$$

Denoting $y \wedge z \rightsquigarrow y'$; interval property:

$$\begin{aligned} \Delta(x \wedge y, y \wedge z) &= \Delta(x \wedge y', y') \\ &\leq \Delta(x, x \vee y') \end{aligned}$$

$x \vee y' \leq_{\mathbb{P}} y$ (since $x \leq_{\mathbb{P}} y$ and $y' \leq_{\mathbb{P}} y$); claim 1:

$$\Delta(x \wedge y, y \wedge z) \leq \Delta(x, y)$$

Functorial maps and their role in constructing GQM

Claim 3: $\Delta(p \wedge q, q) \leq \Delta(p \wedge t, t) * \Delta(t \wedge q, q)$

Proof. With claim 1:

$$\Delta(p \wedge q, q) \leq \Delta(p \wedge t \wedge q, q)$$

Since Δ is functorial:

$$\Delta(p \wedge t \wedge q, q) = \Delta(p \wedge t \wedge q, t \wedge q) * \Delta(t \wedge q, q)$$

With Claim 2:

$$\Delta(p \wedge t \wedge q, t \wedge q) * \Delta(t \wedge q, q) = \Delta(p \wedge t, t) * \Delta(t \wedge q, q)$$

Hence, $\Delta(p \wedge q, q) \leq \Delta(p \wedge t, t) * \Delta(t \wedge q, q)$.

Corollary: D^+ is triangular I

Input: O a set, a finite chain $S = \{0, \ldots, n\}$, ratings $A, B, C : O \to S$. Then:

- $D^+(A,B) \ \leq \ D^+(A,C) \ + \ D^+(C,B)$

The set $\mathcal{O}$ of all ratings $O : A \to S$ is endowed with a natural order:

- $A \leq_{\mathbb{O}} B$ if $A(o) \leq B(o)$ for all $o \in O$.

We write $\mathbb{O} = (\mathcal{O}, \leq_{\mathbb{O}})$. $\mathbb{O}$ is a lattice where

- $(A \vee B)(o) = \max(A(o), B(o))$
- $(A \wedge B)(o) = \min(A(o), B(o))$

Functorial maps and their role in constructing GQM

Theorem (Submodular case)

Let $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}})$ be a lattice. If a map $\Delta \colon \leq_{\mathbb{P}} \longrightarrow M$ is weakly positive, submodular and functorial w. r. t. $(\mathbb{P}, \mathcal{M})$, then

$$\boxed{d \colon P \times P \longrightarrow M, \quad (p, q) \mapsto \Delta(p, p \vee q)}$$

is a GQM w. r. t. $(\mathbb{P}, \mathcal{M})$.